Barcelona GSE
EPP-COMP Brush-up courses 2020-21
Problem Set 3

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Below you will find the collection of exercises meant to refresh the basic concepts of calculus such as limits, derivatives, integrals, power series,etc. Most of exercises provided are at the level one would expect for a graduate student. The importance of these concepts will be reveal to you during the microeconomic, macroeconomic and econometric courses that you will take.

The problem set will be due to 15 September and should be uploaded to the corporate platform by noon.

For handing in the problem set, **YOU CAN:**

- **SCAN** your solutions and send them to me
- Print or Insert images in your solutions

YOU CANNOT:

- have an illegible handwriting if you are to give a hard copy
- send photos of you solutions

1 LIMITS

Problem 1. Solve the following limits

(a) $\lim_{x\to 0} \frac{x(1-x)}{3x^2}$

(b) $\lim_{x\to 0} \frac{x-3}{x^2+x-12}$

(c) $\lim_{x\to 1} \frac{x+2}{x^2-4}$

(d) $\lim_{x\to -1} \frac{2x^2+x-6}{x+2}$

(e) $\lim_{x\to \infty} \frac{2x^3-x^2+7x-3}{2-x+5x^2-4x^3}$

(f) $\lim_{x\to \infty} \sqrt{x+1} - \sqrt{x}$

2 DERIVATIVES

Problem 2. For what value of b does the function $y = x^2 + bx + 1$ have a horizontal tangent at $x = 3$?

Problem 3. Find the two points on the curve $y = x - \frac{1}{4}x^2$ at which the tangent passes through the point $\left(\frac{9}{2}, 0\right)$

Problem 4. Use the three-step rule to calculate $f'(x)$ if $f(x)$ is equal to:

(a) $\frac{x+1}{x}$

(b) $\frac{3-2x}{x-2}$

(c) $\sqrt{3x+2}$

(d) $\sqrt{x^2+1}$

Problem 5. A car riding company takes passengers from one side of the city to the other at a fix price per meter m. The costs of the company vary according to the following **Total Cost Function:** $CT(m) = 3m^2 + 5m + 2$. At what rate is the cost changing at every moment? *Hint: find the rate of change as* $\Delta m \to 0$

Problem 6. Find all points on the curve $y = \frac{6}{x}$ where the tangent is parallel to the line $2x + 3y + 1 = 0$

Problem 7. Sketch the graph of the curve $y = \frac{x}{x+1}$. how many tangent lines pass through the point $(1, 3)$? Find the x-coordinates of the points of tangency of these lines.

Problem 8. Let P be a point on the first-quadrant part of the curve $y = \frac{1}{x}$. Show that the triangle determined by the x-axis, the tangent at P, and the line from P to the origin is isosceles, and find its area.

Problem 9. Find $\frac{\partial s}{\partial t}$:

(a) $s = \frac{1}{(2t-1)^2}$ (b) $s = \frac{t^4-10t^2}{(t^2-6)^2}$

Problem 10. Find $\frac{\partial y}{\partial x}$ by two methods, first without the power rule and then using the power rule.

(a) $y = u^2, u = x^2 - 3x + 2$ (b) $y = u^3, u = x - \frac{1}{x}$

Problem 11. Find $\frac{\partial y}{\partial x}$ by implicit differentiation and also by solving for y and then differentiating, and verify that your two answers are equivalent.

(a) $2x^2 + 3x + y^2 = 12$ (b) $\frac{1-y}{1+y} = x$

Problem 12. Find the tangent line of:

(a) $y = (5-3x)^{\frac{1}{3}}$ at $(-1, 2)$ (b) $x^4 + 16y^4 = 32$ at $(2, 1)$

Problem 13. Find the first four derivatives of:

(a) $8x - 3$

(b) $8x^2 - 11x + 2$

(c) $x^4 - 13x^3 + 5x^2 - 3x - 2$

(d) $x^{\frac{5}{2}}$

3 APPLICATIONS OF DERIVATIVES

Problem 14. Sketch the graphs of the following functions by using the first derivative; in particular find the intervals on which each function is increasing and those on which it is decreasing, and critical points. Also use the second derivative to find where the function is concave up or concave down and use these facts to sort out critical points i.e. if they are maxima, minima or inflection point.

(a) $f(x) = 2x^3 + 3x^2 - 12x$

(b) $f(x) = \frac{9}{x^2+9}$

(c) $f(x) = x - \frac{1}{x}$

Problem 15. For the cost and price functions given, find the production level that maximises profit.

- **Cost function:** $C(x) = 750 + 140x - 0.2x^2 + \frac{1}{30}x^3$
- **Revenue function:** $P(x) = 80 - 0.01x$

Problem 16. Find the linear approximation of $f(x) = \frac{1}{x}$ at $x = -1$

4 INTEGRALS

Problem 17. Compute the integrals. Be sure to include the constant of integration on each answer.

(a) $\int 4x^5 + 6x - 5dx$ (b) $\int x^2\sqrt{x}dx$

Problem 18. Compute the indefinite integrals by the method of substitution.

(a) $\int \sqrt{3-2x}dx$ (b) $\int \frac{4x}{\sqrt{x^2+1}}dx$

Problem 19. Compute the definite integrals by the method of substitution. Do not forget about changing the limits of integration.

(a) $\int_1^2 \frac{(2x+1)dx}{\sqrt{x^2+x+2}}$ (b) $\int_1^e \frac{\sqrt{\ln x}dx}{x}dx$

Problem 20. Compute the definite integrals by parts. Do not forget about computing the limits of integration for each part.

(a) $\int_1^2 \ln x$ (b) $\int_0^1 x^2e^x dx$

5 POWER SERIES

Problem 21. Find the sum of the series:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + ... = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Hint: derive the expanded expression, i.e. the term at the left hand side, then see if it looks familiar from class and take the integral

Problem 22. Use the formula given in the slides to find the Taylor's series of each of the following functions:

(a) $y = e^{-x}$ (c) $y = \ln(1+x)$

(b) $y = e^{3x}$ (d) $y = \frac{1}{(1+x)}$

6 MULTIVARIATE CALCULUS

Problem 23. Find the domain of the following functions and express it in set notation; sketch some level curves for those functions for which is feasible:

(a) $f(x,y) = \frac{xy}{y-2x}$

(b) $f(x,y) = \frac{1}{x} + \frac{1}{y}$

(c) $f(x,y) = \frac{1}{(e^x+e^y)^2}$

(d) $f(x,y) = \ln y - 3x$

(e) $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

(f) $f(x,y,z) = \frac{1}{\sqrt{16-x^2-y^2-z^2}}$

Problem 24. Find the partial derivatives w.r.t. x, y, and z; also find the cross derivatives and verify that $f_{xy} = f_{yx}$ for all combinations:

(a) $f(x,y,z) = x^2y^5z^7$

(b) $f(x,y,z) = x \ln \frac{y}{z}$

(c) $f(x,y,z) = e^{x^2+y^3+z^4}$

7 IMPLICIT FUNCTION THEOREM

Problem 25. Make use of the implicit function theorem seen in class to derive $\frac{dy}{dx}$ of the following expressions.

(a) $x^6 + 2y^4 = 1$

(b) $e^{xy} = 2xy^2$

Problem 26. Make use of the implicit function theorem seen in class to derive $\frac{dz}{dx}$ and $\frac{dz}{dy}$ of the expression $lnz = z + 2x - 3y$.

Problem 27. If $F(x,y)$ has continuous second partial derivatives and the equation $F(x,y) = c$ defines $y = f(x)$ as a twice-differentiable function, show that for $F_y(x,y) \neq 0$,

$$\frac{\partial^2 y}{\partial x^2} = \frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_x^2F_{yy}}{F_y^3}$$

8 CONCAVITY AND CONVEXITY

Problem 28. Use the segment test and the first derivative test to check the concavity(convexity) of the following functions

(a) $z = x^2$ (b) $z = x_1^2 + x_2^2$ (c) $z = 2x^2 - xy + y^2$